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INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 10625–10638 PII: S0305-4470(04)79856-X

Resonant triads for two bidirectional equations in 1 + 1 dimensions

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Received 27 April 2004, in final form 17 September 2004
Published 20 October 2004
Online at stacks.iop.org/JPhysA/37/10625
doi:10.1088/0305-4470/37/44/011

Abstract

The two-soliton solution of the KdV equation is known not to degenerate into a solution describing resonant triads of solitons in 1 + 1 dimensions. However, we show that two integrable coupled-KdV systems of the Drinfeld–Sokolov class possess solutions which may degenerate into resonant triads. These solutions are associated with a resonance relation which generalizes the usual one previously considered by Hirota and Ito.

PACS numbers: 02.30.Jr, 05.45.Yv

1. Introduction

Solitons are often known to be stable and to interact elastically, such that they emerge from the interaction without changing their forms. However, when a 'resonant' relation between the parameters of the soliton solution is satisfied, integrable PDEs may describe structural instabilities such as the decay of a solitary wave into two new solitary waves in a finite time or the fusion of two waves into a single one. The existence of such a triad results from the degeneracy of the two-soliton solution when the coupling factor becomes zero or infinite. It is well known that in the KdV case, this coupling factor or its inverse never vanishes. However, resonant interactions in 1 + 1 dimensions may exist and have been mostly discussed [1–5] for soliton equations possessing solitary waves of the KdV type

$$u = \partial_x^2 \log(1 + e^{\theta}) \simeq \operatorname{sech}^2 \frac{\theta}{2} \tag{1}$$

or of the form of a kink [6, 7] on a nonvanishing background.

The integrable PDEs of fifth order such as Sawada–Kotera (SK) and Kaup–Kupershmidt (KK) have attracted special attention since the discovery [8] of their link with some integrable cases of the cubic Hénon–Heiles Hamiltonian, and also for their ability to describe resonant interactions of solitons. It is known that the SK and KK equations possess soliton solutions describing the elastic collision of sech^2-type waves for SK and of a more general type for KK

$$u = \partial_x^2 \log(1 + \mathrm{e}^{\theta} + \mathrm{e}^{2\theta}). \tag{2}$$

The soliton solution of SK, built on a nonvanishing background, is known to degenerate when the coupling factor (or its inverse) becomes zero into a regular solution describing inelastic collisions of solitons. However, as shown in section 2, the same conditions on the coupling factor for KK, implies that the degenerate two-soliton becomes singular for finite values of x and t.

The goal of the present paper is to investigate the behaviour of the two-soliton solution of other integrable PDEs possessing solitary waves of both types (1) and (2), in order to prove the existence of resonant triads involving solitary waves of the KdV and KK type. In [9], we studied the soliton solutions of two coupled KdV systems from the list of Drinfeld and Sokolov [10]:

$$u_t = \frac{1}{2}u_{xxx} + auu_x + 3w_x, \tag{3}$$

$$w_t = -w_{xxx} - auw_x, \tag{4}$$

$$u_t = -\frac{35}{2}u_{xxx} - 80auu_x + 5v_x, \tag{5}$$

$$v_t = \frac{25}{2}v_{xxx} - \frac{171}{4}u_{xxxxx} - 20a(15uu_{xxx} + 18u_xu_{xx} + 2u_xv - 2uv_x), \tag{6}$$

with a an arbitary constant. We were interested in these systems for their link with some integrable cases of the quartic Hénon–Heiles Hamiltonian [11, 12] as well as for the fact that they can describe overtaking and head-on collisions of solitons. We showed that the system (3)–(4) possess two different types of solitary waves according to the direction of propagation (KdV or KK type), while the system (5)–(6) possess only KK-type solitary waves, but their profiles depend on the direction of propagation.

In section 3, we establish that the two-soliton solution associated with the system (5)–(6) may display triads of resonant solitons only when the solutions are built on a nonvanishing background. We show that the three waves involved in the fusion or in the decay process have the same profile, independently of their direction of propagation.

In section 4, we establish that the two-soliton of the system (3)–(4) built on a nonvanishing background can also degenerate into a resonant triad involving two waves of the KdV type and one of the KK type. Moreover, in this case, a new type of resonance relation occurs, extending the one previously considered in [5].

2. The KK equation

In this section, we show that, for the KK equation, the condition for becoming zero or infinite on the coupling factor of the soliton solution implies that this solution becomes singular at finite distance and time.

The N-soliton solution of the KK equation

$$u_t + \left(u_{xxxx} + 40auu_{xx} + 30au_x^2 + \frac{320}{3}a^2u^3\right)_x = 0 \tag{7}$$

on a constant background c is given by

$$u_N = \frac{3}{8a}\partial_x^2 \log f_N + c. \tag{8}$$

The function f_N has the form of a Grammian [13]

$$f_N = \det\left[\int \psi_i \psi_j \,\mathrm{d}x\right]_{1\leqslant i,j\leqslant N}, \tag{9}$$

where ψ_i satisfies the third-order Lax pair

$$\lambda\psi = \left(\partial_x^3 + 8au\partial_x + 4au_x\right)\psi, \tag{10}$$

$$\partial_t\psi = \left(9\lambda\partial_x^2 - (4au_{xx} + 64a^2u^2)\partial_x + 4au_{xxx} + 128a^2uu_x + 48a\lambda u\right)\psi, \tag{11}$$

with $u = c$ and $\lambda = \lambda_i$. Since a is a normalization constant, we further take $a = 1$ and obtain the solitary wave

$$\begin{aligned} &u_1(k) = \frac{3}{8}\partial_x^2 \log(1 + 4\mathrm{e}^{\theta} + A(k)\mathrm{e}^{2\theta}) + c, \qquad A(k) = \frac{k^2 + 32c}{k^2 + 8c}, \\ &\theta = kx - \omega(k)t + \delta, \qquad \omega(k) = k^5 + 40ck^3 + 320c^2k. \end{aligned} \tag{12}$$

The two-soliton solution is given by

$$\begin{aligned} u_2(k_1, k_2) &= \tfrac{3}{8}\partial_x^2 \log f_2(k_1, k_2) + c, \\ f_2(k_1, k_2) &= 1 + 4\mathrm{e}^{\theta_1} + 4\mathrm{e}^{\theta_2} + A_1\mathrm{e}^{2\theta_1} + A_2\mathrm{e}^{2\theta_2} + 8B_{12}\mathrm{e}^{\theta_1+\theta_2} \\ &\quad + 4A_{12}\mathrm{e}^{\theta_1+\theta_2}(A_1\mathrm{e}^{\theta_1} + A_2\mathrm{e}^{\theta_2}) + A_1A_2A_{12}^2\,\mathrm{e}^{2(\theta_1+\theta_2)}, \end{aligned} \tag{13}$$

$$A_i = A(k_i), \quad \theta_i = \theta(k_i), \qquad A_{12} = \frac{(k_1 - k_2)^2\left(k_1^2 - k_1k_2 + k_2^2 + 24c\right)}{(k_1 + k_2)^2\left(k_1^2 + k_1k_2 + k_2^2 + 24c\right)}, \tag{14}$$

$$B_{12} = \frac{2(k_1 + k_2)^4 - k_1^2k_2^2 + 48c\left(k_1^2 + k_2^2\right)}{(k_1 + k_2)^2\left(k_1^2 + k_1k_2 + k_2^2 + 24c\right)}. \tag{15}$$

A two-soliton solution can degenerate into a resonant triad under the conditions

$$A_{12} = 0 \quad \text{or} \quad (A_{12})^{-1} = 0, \tag{16}$$

implying the resonance relation [5]

$$\omega(k_1) \mp \omega(k_2) = \omega(k_1 \mp k_2). \tag{17}$$

We first consider the case $A_{12} = 0$, which corresponds to

$$E \equiv k_1^2 - k_1k_2 + k_2^2 + 24c = 0 \tag{18}$$

and possesses two solutions for k_2

$$k_2 \equiv k_2^{\pm} = \frac{k_1}{2} \pm \frac{1}{2}\sqrt{-3k_1^2 - 96c}, \tag{19}$$

which are only real for $c < 0$ and $k_1^2 < -32c$.

In order to have a regular solution, the conditions

$$A_1 > 0, \quad A_2 > 0 \quad \text{and} \quad B_{12} > 0$$

must be satisfied. $A_1 > 0$ for $|k_1| < \sqrt{-8c}$. For $k_2 = k_2^+$, $A_2 > 0$ for $k_1 < -\sqrt{-8c}$, while for $k_2 = k_2^-$, $A_2 > 0$ for $k_1 > \sqrt{-8c}$, such that it is impossible to have simultaneously

$$A_{12} = 0, \quad A_1 > 0, \quad A_2 > 0 \quad \text{and} \quad B_{12} > 0.$$

An analogous analysis for the case $(A_{12})^{-1}$ gives the same result. We therefore conclude that, when the resonance relation (17) is satisfied, the two-soliton solution of KK becomes singular at finite distance and time, and cannot describe the process of fusion or decay involving three waves of type (2).

3. The bKK equation

By elimination of the field v, the system (5)–(6) are equivalent with the soliton equation

$$z_{tt} + 5\left(z_{xx,t} + 8az_x z_t - z_{xxxxx} - 40az_x z_{xxx} - 30a(z_{xx})^2 - \frac{320}{3}a^2 z_x^3\right)_x = 0, \quad (20)$$

called the bidirectional Kaup–Kupershmidt equation (bKK) in [14, 9].

In this section we show that, contrary to the KK equation, the bKK equation may possess regular solutions describing the resonant interactions of three waves of type (2) with the same bell-shaped profile.

Extending the results previously obtained in [9], we consider the N-soliton solution of equation (20) on a nonvanishing constant background c

$$u_N = (z_N)_x = \frac{3}{8a}\partial_x^2 \log f_N + c, \quad (21)$$

with f_N written in the form of a Grammian as in (9), but where the functions ψ_i satisfy the fifth-order Lax pair

$$\lambda\psi = \left(\partial_x^5 + \tfrac{40}{3}az_x\partial_x^3 + 20az_{xx}\partial_x^2 + \tfrac{1}{9}\left(320a^2z_x^2 + 140az_{xxx} + 8az_t\right)\partial_x + \tfrac{1}{9}(320a^2z_xz_{xx} + 40az_{xxxx} + 4az_{xt})\right)\psi, \quad (22)$$

$$\partial_t\psi = 5\left(\partial_x^3 + 8az_x\partial_x + 4az_{xx}\right)\psi, \quad (23)$$

with $z = cx$ and $\lambda \equiv \lambda_i$.

If we apply the translation $z = \tilde{z} + cx$ on equation (20) and setting $a = 1$, we have that the dispersion relation for the equation in $\tilde{z}$ is given by

$$F(\omega, k) \equiv \omega^2 + 5(\omega k^3 + 8c\omega k - k^6 - 40ck^4 - 320c^2k^2) = 0, \quad (24)$$

which is of second degree in ω. We therefore may define, as in [9], two different speeds:

$$v_\pm(k) = \frac{5}{2}k^2 + 20c \pm \frac{\sqrt{5}}{2}(3k^2 + 40c),$$

with '+', '−' corresponding to the direction of propagation for $c = 0$.

The solitary waves are

$$u_1^{(\pm)}(k) = \tfrac{3}{8}\partial_x^2 \log(1 + 4e^{\theta_\pm} + A^{(\pm)}(k)e^{2\theta_\pm}) + c, \quad (25)$$

$$A^{(\pm)}(k) = \frac{3k^2(5 \pm \sqrt{5}) + 320c}{3k^2(5 \mp \sqrt{5}) + 80c}, \qquad \theta_\pm = kx - kv_\pm t + \delta_\pm. \quad (26)$$

We stated in [9] that, for $c = 0$, the solitary wave propagating in the negative direction possesses a unusual shape (see figure 1(b)), different from the profile of the bell-shaped wave propagating in the positive direction (see figure 1(a)). For $c \neq 0$, we here show that the profile of the solitary wave $u_1^{(-)}$ may change, according to the value of k in function of c. In order to determine if the wave $u_1^{(-)}$ may possess a profile as in figure 1(b), we look for the condition on k for the existence of three real extrema. The result of this analysis is that $u_1^{(-)}$ possesses three real extrema if $c < 0$, for

$$|k| > k_{\min} \quad (27)$$

$$k_{\min} = \frac{4}{3}\sqrt{-3(5 + \sqrt{5})c}, \quad (28)$$

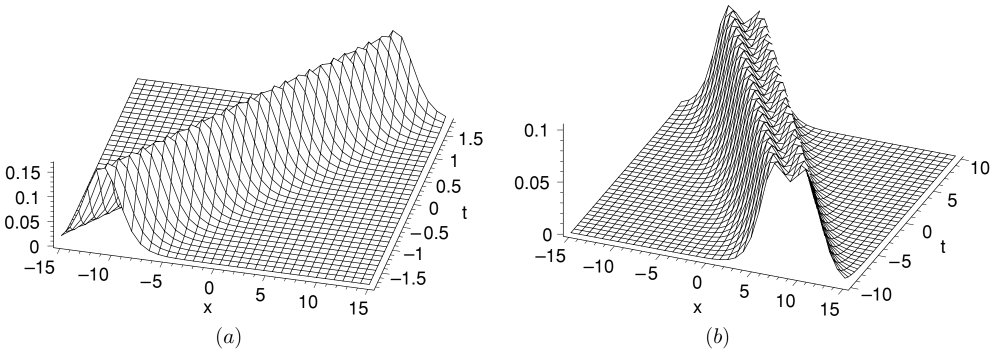

Figure 1. For $c = 0$, solitary waves of equation (20): (*a*) $v_+ > 0$, (*b*) $v_- < 0$.

if $c > 0$, for

$$|k| \geqslant \frac{2}{5}\sqrt{50\sqrt{5}c}. \tag{29}$$

At the level of the two-soliton solution, we have three possibilities:

$$u_2^{(\pm\pm)} = \tfrac{3}{8}\partial_x^2 \log f_2^{(\pm\pm)} + c, \tag{30}$$

$$\begin{aligned} f_2^{(\pm\pm)}(k_1, k_2) = 1 &+ 4\mathrm{e}^{\theta_{1,\pm}} + 4\mathrm{e}^{\theta_{2,\pm}} + 8B_{12}^{(\pm\pm)}\mathrm{e}^{\theta_{1,\pm}+\theta_{2,\pm}} \\ &+ A_1^{(\pm)}\mathrm{e}^{2\theta_{1,\pm}} + A_2^{(\pm)}\mathrm{e}^{2\theta_{2,\pm}} + 4A_{12}^{(\pm\pm)}\mathrm{e}^{\theta_{1,\pm}+\theta_{2,\pm}}\left(A_1^{(\pm)}\mathrm{e}^{\theta_{1,\pm}} + A_2^{(\pm)}\mathrm{e}^{\theta_{2,\pm}}\right) \\ &+ A_1^{(\pm)}A_2^{(\pm)}\left(A_{12}^{(\pm\pm)}\right)^2\mathrm{e}^{2(\theta_{1,\pm}+\theta_{2,\pm})}, \qquad A_i^{(\pm)} = A^{(\pm)}(k_i), \end{aligned} \tag{31}$$

$$B_{12}^{(\pm\pm)} = \frac{12\left(k_1^4 + k_2^4\right) - 3(3 \mp \sqrt{5})k_1^2k_2^2 + 160c\left(k_1^2 + k_2^2\right)}{\left(6\left(k_1^2 + k_2^2\right) + 3(1 \mp \sqrt{5})k_1k_2 + 80c\right)(k_1 + k_2)^2}, \tag{32}$$

$$A_{12}^{(\pm\pm)} = \frac{\left(6\left(k_1^2 + k_2^2\right) - 3(1 \mp \sqrt{5})k_1k_2 + 80c\right)(k_1 - k_2)^2}{\left(6\left(k_1^2 + k_2^2\right) + 3(1 \mp \sqrt{5})k_1k_2 + 80c\right)(k_1 + k_2)^2}, \tag{33}$$

$$u_2^{(+-)} = \tfrac{3}{8}\partial_x^2 \log f_2^{(+-)}, \tag{34}$$

$$\begin{aligned} f_2^{(+-)}(k_1, k_2) = 1 &+ 4\mathrm{e}^{\theta_{1,+}} + 4\mathrm{e}^{\theta_{2,-}} + 32B_{12}^{(+-)}\mathrm{e}^{\theta_{1,+}+\theta_{2,-}} + A_1^{(+)}\mathrm{e}^{2\theta_{1,+}} \\ &+ A_2^{(-)}\mathrm{e}^{2\theta_{2,-}} + 4A_{12}^{(+-)}\mathrm{e}^{\theta_{1,+}+\theta_{2,-}}\left(A_1^{(+)}\mathrm{e}^{\theta_{1,+}} + A_2^{(-)}\mathrm{e}^{\theta_{2,+}}\right) \\ &+ A_1^{(+)}A_2^{(-)}\left(A_{12}^{(+-)}\right)^2\mathrm{e}^{2(\theta_{1,+}+\theta_{2,-})}, \end{aligned} \tag{35}$$

$$B_{12}^{(+-)} = \frac{27\left(k_1^4 + k_2^4\right) + 9\sqrt{5}\left(k_1^4 - k_2^4\right) + 9k_1^2k_2^2 + 40c\left(15\left(k_1^2 + k_2^2\right) + 3\sqrt{5}\left(k_1^2 - k_2^2\right) + 80c\right)}{\left(6k_1^2 + 3(3 - \sqrt{5})(k_1 + k_2)k_2 + 80c\right)\left(6k_2^2 + 3(3 + \sqrt{5})(k_1 + k_2)k_1 + 80c\right)}, \tag{36}$$

$$A_{12}^{(+-)} = \frac{\left(6k_1^2 - 3(3 - \sqrt{5})(k_1 - k_2)k_2 + 80c\right)\left(6k_2^2 + 3(3 + \sqrt{5})(k_1 - k_2)k_1 + 80c\right)}{\left(6k_1^2 + 3(3 - \sqrt{5})(k_1 + k_2)k_2 + 80c\right)\left(6k_2^2 + 3(3 + \sqrt{5})(k_1 + k_2)k_1 + 80c\right)}. \tag{37}$$

Among those solutions, only $u_2^{(++)}$ can degenerate into a solution describing a resonant interaction. Indeed, the function f_2 has no zeros if $A_i > 0, i = 1, 2$ and $B_{12} > 0$ and we observe that the simultaneous conditions $A_{12}^{(--)} = 0$ (or $(A_{12}^{(--)})^{-1} = 0$) and $A_i^{(-)} > 0$, $i = 1, 2$ or $A_{12}^{(+-)} = 0$ (or $(A_{12}^{(+-)})^{-1} = 0$), $A_1^{(+)} > 0$, $A_2^{(-)} > 0$ cannot be realized.

We therefore study the behaviour of the two-soliton solution in the two degenerate cases $A_{12}^{(++)} = 0$ or $\left(A_{12}^{(++)}\right)^{-1} = 0$. The first case implies the relation

$$\omega_+(k_1) - \omega_+(k_2) = \omega_-(k_1 - k_2). \tag{38}$$

Although this relation is different from the usual resonant relation (17), we will show that it corresponds to a resonant interaction involving three bell-shaped waves.

For $c < 0$, the equation

$$E_- \equiv 6\left(k_1^2 + k_2^2\right) - 3(1 - \sqrt{5})k_1k_2 + 80c = 0, \tag{39}$$

corresponds, in the real plane (k_1, k_2), to an ellipse. The two solutions of this equation are given by

$$k_2^{\pm} = \frac{1-\sqrt{5}}{4}k_1 \pm \frac{1}{12}\sqrt{-18(5+\sqrt{5})k_1^2 - 1920c}, \qquad k_1^2 \leqslant -\frac{16(5-\sqrt{5})}{3}c, \tag{40}$$

and in this case expression (31) degenerates into

$$f_2^{(++)}(k_1, k_2) = 1 + 4e^{\theta_{1,+}} + 4e^{\theta_{2,+}} + 8B_{12}^{(++)}e^{\theta_{1,+}+\theta_{2,+}} + A_1^{(+)}e^{2\theta_{1,+}} + A_2^{(+)}e^{2\theta_{2,+}}. \tag{41}$$

The condition $A_i^{(+)} > 0$, $i = 1, 2$ implies the restriction on k_1

$$\frac{2}{3}\sqrt{-6(5-2\sqrt{5})c} < k_1 < \frac{k_{\min}}{2}, \qquad \text{for} \quad k_2 = k_2^+, \tag{42}$$

$$-\frac{k_{\min}}{2} < k_1 < -\frac{2}{3}\sqrt{-6(5-2\sqrt{5})c}, \qquad \text{for} \quad k_2 = k_2^-, \tag{43}$$

such that, as illustrated in figure 2, only parts of the ellipse can be considered. This guarantees that $B_{12}^{(++)} > 0$ and excludes the possibility, in the resonance process, of involving a wave possessing the profile of figure 1(*b*).

We first consider the case $k_2 = k_2^+$. The speeds which are involved in the resonance relation (38) are

$$v_+(k_1) = \frac{5+3\sqrt{5}}{2}k_1^2 + 20(1+\sqrt{5})c, \tag{44}$$

$$v_+\left(k_2^+\right) = -\frac{5+2\sqrt{5}}{2}k_1^2 - \frac{5+\sqrt{5}}{24}k_1\sqrt{-18k_1^2(5+\sqrt{5}) - 1920c} - \frac{40}{3}c, \tag{45}$$

$$v_-\left(k_1 - k_2^+\right) = \frac{\sqrt{5}-5}{4}k_1^2 + \frac{\sqrt{5}}{12}k_1\sqrt{-18k_1^2(5+\sqrt{5}) - 1920c} - \frac{40}{3}c. \tag{46}$$

Those speeds satisfy the inequality

$$v_+(k_1) < v_+\left(k_2^+\right) < v_-\left(k_1 - k_2^+\right), \qquad \text{for } k_1 < \frac{2}{3}\sqrt{-15(3-\sqrt{5})c}, \tag{47}$$

$$v_+\left(k_2^+\right) < v_+(k_1) < v_-\left(k_1 - k_2^+\right), \qquad \text{for } k_1 > \frac{2}{3}\sqrt{-15(3+\sqrt{5})c}. \tag{48}$$

In order to identify the resonant interaction, we are looking for the behaviour of (41) in three particular frames

(i) $\theta_{1,+}$ fixed, $x = \xi + v_+(k_1)t$,

$$\begin{aligned}\theta_{2,+} &= k_2^+ x - k_2^+ v_+\big(k_2^+\big)t + \delta_{2,+}\\ &= k_2^+\xi + k_2^+\big(v_+(k_1) - v_+\big(k_2^+\big)\big)t + \delta_{2,+}\\ &\to -\infty \quad \text{for} \quad t \to +\infty \quad \text{when} \quad k_1 < \frac{2}{3}\sqrt{-15(3-\sqrt{5})c},\\ &\to -\infty \quad \text{for} \quad t \to -\infty \quad \text{when} \quad k_1 > \frac{2}{3}\sqrt{-15(3-\sqrt{5})c},\end{aligned}$$

therefore

$$f_2^{(++)}\big(k_1, k_2^+\big) \to 1 + 4\mathrm{e}^{\theta_{1,+}} + A_1^{(+)}\mathrm{e}^{2\theta_{1,+}} \tag{49}$$

for $t \to +\infty$ when $k_1 < \frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$ and for $t \to -\infty$ when $\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$.

(ii) $\theta_{2,+}$ fixed, $x = \xi + v_+(k_2^+)t$,

$$\begin{aligned}\theta_{1,+} &= k_1\xi + k_1\big(v_+\big(k_2^+\big) - v_+(k_1)\big)t + \delta_{1,+}\\ &\to -\infty \quad \text{for} \quad t \to -\infty \quad \text{when} \quad k_1 < \frac{2}{3}\sqrt{-15(3-\sqrt{5})c}\\ &\to -\infty \quad \text{for} \quad t \to +\infty \quad \text{when} \quad k_1 > \frac{2}{3}\sqrt{-15(3-\sqrt{5})c}\end{aligned}$$

therefore

$$f_2^{(++)}\big(k_1, k_2^+\big) \to 1 + 4\mathrm{e}^{\theta_{2,+}} + A_2^{(+)}\mathrm{e}^{2\theta_{2,+}} \tag{50}$$

for $t \to -\infty$ when $k_1 < \frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$ and for $t \to +\infty$ when $k_1 > \frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$.

(iii) $\theta_{1,+} - \theta_{2,+}$ fixed, $x = \xi + v_-\big(k_1 - k_2^+\big)t$,

$$\begin{aligned}\theta_{1,+} &= k_1\xi + k_1\big(v_-\big(k_1 - k_2^+\big) - v_+(k_1)\big)t + \delta_{1,+} \to +\infty \text{ for } t \to +\infty,\\ \theta_{2,+} &= k_2^+\xi + k_2^+\big(v_-\big(k_1 - k_2^+\big) - v_+\big(k_2^+\big)\big)t + \delta_{2,+} \to +\infty \text{ for } t \to +\infty,\end{aligned}$$

therefore

$$f_2^{(++)}\big(k_1, k_2^+\big) \to A_2^{(+)}\mathrm{e}^{2\theta_{2,+}}\big(1 + 4\mathrm{e}^{\theta_{1,+}-\theta_{2,+}} + A^{(+)}\big(k_1 - k_2^+\big)\mathrm{e}^{2(\theta_{1,+}-\theta_{2,+})}\big) \quad \text{for} \quad t \to +\infty, \tag{51}$$

where $\log 2B_{12}^{(++)}$ is inserted in the phase.

Therefore, for $k_2 = k_2^+$, the degenerate two-soliton solution describes the decay of a solitary wave into two new waves (see figure 3(*a*)).

Let us now consider the case $k_2 = k_2^-$. Because of the restriction on k_1, we have that $k_1 < 0$, $k_2^- < 0$, such that the speeds satisfy the inequality

$$v_+(k_2^-) < v_+(k_1) < v_-(k_1 - k_2^-), \qquad \text{for } k_1 < -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}, \tag{52}$$

$$v_+(k_1) < v_+(k_2^-) < v_-(k_1 - k_2^-), \qquad \text{for } k_1 > -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}. \tag{53}$$

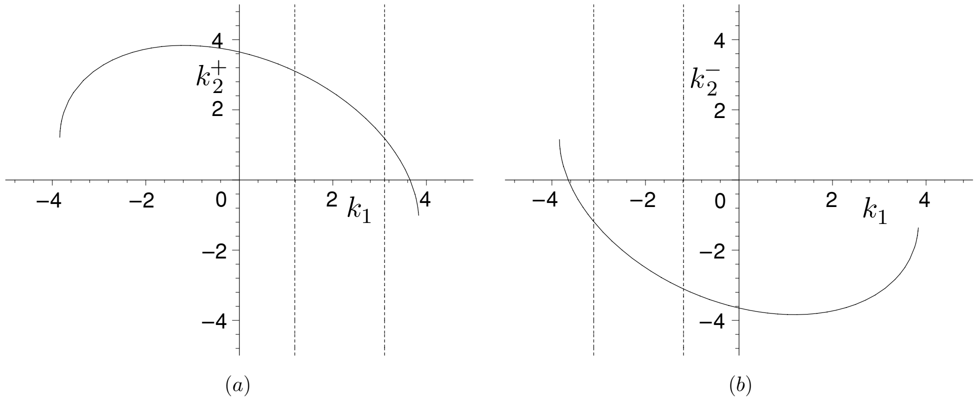

Figure 2. Solid line: k_2^+ (*a*) and k_2^- (*b*) as a function of k_1, dashed lines: lines between which $A_i > 0,\ i = 1, 2$ $(c = -1)$.

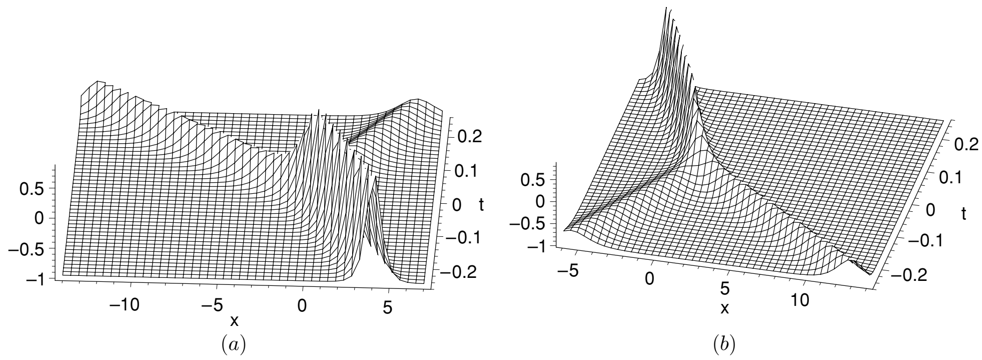

Figure 3. Resonant interaction of bKK solitons, giving rise to decay for $k_1 = 8/5$ (*a*) and fusion for $k_1 = -8/5$ (*b*) of solitary waves $(c = -1)$.

The behaviour of the solution in the three previous frames is

(i) $\theta_{1,+}$ fixed, $x = \xi + v_+(k_1)t$,

$$\begin{aligned}\theta_{2,+} &= k_2^- \xi + k_2^-(v_+(k_1) - v_+(k_2^-))t + \delta_{2,+}\\ &\to -\infty \quad \text{for} \quad t \to +\infty \quad \text{when} \quad k_1 < -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c},\\ &\to -\infty \quad \text{for} \quad t \to -\infty \quad \text{when} \quad k_1 > -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c},\end{aligned}$$

therefore

$$f_2^{(++)}(k_1, k_2^-) \to 1 + 4\mathrm{e}^{\theta_{1,+}} + A_1^{(+)}\mathrm{e}^{2\theta_{1,+}} \tag{54}$$

for $t \to +\infty$ when $k_1 < -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$ and for $t \to -\infty$ when $k_1 > -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$.

(ii) $\theta_{2,+}$ fixed, $x = \xi + v_+(k_2^-)t$,

$$\begin{aligned}\theta_{1,+} &= k_1 \xi + k_1(v_+(k_2^-) - v_+(k_1))t + \delta_{1,+}\\ &\to -\infty \quad \text{for} \quad t \to -\infty \quad \text{when} \quad k_1 < -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}\\ &\to -\infty \quad \text{for} \quad t \to +\infty \quad \text{when} \quad k_1 > -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c},\end{aligned}$$

therefore

$$f_2^{(++)}(k_1, k_2^-) \to 1 + 4e^{\theta_{2,+}} + A_2 e^{2\theta_{2,+}} \tag{55}$$

for $t \to -\infty$ when $k_1 < -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$ and for $t \to +\infty$ when $k_1 > -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}$.

(iii) $\theta_{1,+} - \theta_{2,+}$ fixed, $x = \xi + v_-(k_1 - k_2^-)t$,

$$\theta_{1,+} = k_1\xi + k_1(v_-(k_1 - k_2^-) - v_+(k_1))t + \delta_{1,+} \to +\infty \text{ for } t \to -\infty,$$

$$\theta_{2,+} = k_2^-\xi + k_2^-(v_-(k_1 - k_2^-) - v_+(k_2^-))t + \delta_{2,+} \to +\infty \text{ for } t \to -\infty,$$

therefore

$$f_2^{(++)}(k_1, k_2^-) \to A_2^{(+)} e^{2\theta_{2,+}} \left(1 + 4e^{\theta_{1,+}-\theta_{2,+}} + A^{(+)}(k_1 - k_2^-) e^{2(\theta_{1,+}-\theta_{2,+})}\right)$$
$$\text{for } t \to -\infty. \tag{56}$$

Therefore, for $k_2 = k_2^-$, the degenerate two-soliton solution describes a process of fusion of two solitary waves into a new wave (see figure 3(*b*)).

Let us now consider the implication of the condition $\left(A_{12}^{(++)}\right)^{-1} = 0$), which yields the resonance relation

$$\omega_+(k_1) + \omega_+(k_2) = \omega_-(k_1 + k_2). \tag{57}$$

For $c < 0$, the equation

$$E_+ \equiv 6\left(k_1^2 + k_2^2\right) + 3(1 - \sqrt{5})k_1k_2 + 80c = 0, \tag{58}$$

also represents an ellipse in the real plane (k_1, k_2), and yields the two solutions for k_2

$$k_2 \equiv k_2^\pm = \frac{\sqrt{5}-1}{4}k_1 \pm \frac{1}{12}\sqrt{-18(5+\sqrt{5})k_1^2 - 1920c}. \tag{59}$$

The condition $A_i > 0$, $i = 1, 2$ implies the restriction on k_1

$$-\frac{k_{\min}}{2} < k_1 < -\frac{2}{3}\sqrt{-6(5-2\sqrt{5})c}, \qquad \text{for} \quad k_2 = k_2^+, \tag{60}$$

$$\frac{2}{3}\sqrt{-6(5-2\sqrt{5})c} < k_1 < \frac{k_{\min}}{2}, \qquad \text{for} \quad k_2 = k_2^-. \tag{61}$$

For $k_2 = k_2^+$, the speeds which are involved in relation (38) are

$$v_+(k_1) = \frac{5+3\sqrt{5}}{2}k_1^2 + 20(1+\sqrt{5})c, \tag{62}$$

$$v_+\left(k_2^+\right) = -\frac{5+2\sqrt{5}}{2}k_1^2 + \frac{5+\sqrt{5}}{24}k_1\sqrt{-18(5+\sqrt{5})k_1^2 - 1920c} - \frac{40}{3}c, \tag{63}$$

$$v_-\left(k_1 + k_2^+\right) = \frac{\sqrt{5}-5}{4} - \frac{\sqrt{5}}{12}k_1\sqrt{-18(5+\sqrt{5})k_1^2 - 1920c} - \frac{40}{3}c, \tag{64}$$

and for $k_1 < 0$, $k_2 = k_2^+ > 0$ we have the inequality

$$v_+\left(k_2^+\right) < v_+(k_1) < v_-\left(k_1 + k_2^+\right), \qquad \text{for} \quad k_1 < -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}, \tag{65}$$

$$v_+(k_1) < v_+\left(k_2^+\right) < v) - \left(k_1 + k_2^+\right), \qquad \text{for} \quad k_1 > -\frac{2}{3}\sqrt{-15(3-\sqrt{5})c}. \tag{66}$$

In order to obtain a nontrivial solution, the limit $A_{12} \to \infty$ can be considered in three different ways:

(i) Choosing the phase $\delta_{1,+} = -\log A_{12}^{(++)}$, expression (31) becomes

$$f_2^{(++)} = 1 + 4\mathrm{e}^{\theta_{2,+}} + 8\frac{B_{12}^{(++)}}{A_{12}^{(++)}}\,\mathrm{e}^{\theta_{1,+}+\theta_{2,+}} + A_2^{(+)}\,\mathrm{e}^{2\theta_{2,+}} + 4A_2^{(+)}\,\mathrm{e}^{\theta_{1,+}+2\theta_{2,+}} + A_1^{(+)}A_2^{(+)}\,\mathrm{e}^{2(\theta_{1,+}+\theta_{2,+})}, \tag{67}$$

and $u_2^{(++)}$ describes the fusion of two solitary waves into a new one.

(ii) Choosing the phase $\delta_{2,+} = -\log A_{12}^{(++)}$, expression (31) becomes

$$f_1^{(++)} = 1 + 4\mathrm{e}^{\theta_{1,+}} + 8\frac{B_{12}^{(++)}}{A_{12}^{(++)}}\,\mathrm{e}^{\theta_{1,+}+\theta_{2,+}} + A_1^{(+)}\,\mathrm{e}^{2\theta_{1,+}} + 4A_1^{(+)}\,\mathrm{e}^{2\theta_{1,+}+\theta_{2,+}} + A_1^{(+)}A_2^{(+)}\,\mathrm{e}^{2(\theta_{1,+}+\theta_{2,+})}, \tag{68}$$

and leads to the inverse process of decay.

(iii) Choosing the phases $\delta_{1,+} = \delta_{2,+} = -\frac{1}{2}\log A_{12}^{(++)}$, expression (31) becomes

$$f_2^{(++)} = 1 + 8\frac{B_{12}^{(++)}}{A_{12}^{(++)}}\,\mathrm{e}^{\theta_{1,+}+\theta_{2,+}} + A_1^{(+)}A_2^{(+)}\,\mathrm{e}^{2(\theta_{1,+}+\theta_{2,+})}, \tag{69}$$

and $u_2^{(++)}$ degenerates into a single-solitary wave with speed $v_-\big(k_1 + k_2^+\big)$.

For $k_2 = k_2^-$ one can prove that the three previous kinds of processes may also occur.

In conclusion, we show that the degeneracy of the two-soliton solution on ellipses (39) and (58) yields a regular solution only on some parts of those curves. In this case, the asymptotic analysis shows that the resonant triad can describe a fusion or a decay process involving three waves of the same bell-shaped profile as in figure 1(*a*).

4. The bSH equation

In this section, we analyse the two-soliton of the system (3)–(4) in order to obtain the conditions for its degeneracy into a resonant triad involving bell-shaped waves of the KdV and KK type.

By elimination of w, the system (3)–(4) are equivalent to the sixth-order soliton equation

$$2z_{tt} + \big(z_{xx,t} - z_{xxxxx} - 3a(z_x z_{xx})_x + \tfrac{3}{2}a(z_{xx})^2 - \tfrac{2}{3}a^2 z_x^3\big)_x = 0, \tag{70}$$

that we called bidirectional Satsuma–Hirota equation in [9].

Extending the results obtained in [9], we consider the N-soliton solutions of equation (70) on a nonvanishing constant background c

$$u_N = (z_N)_x = \frac{6}{a}\partial_x^2 \log f_N + c. \tag{71}$$

The function f_N has, as for the KK equation, the form of a Grammian (9), but where the functions ψ_i satisfy the fourth-order Lax pair

$$\lambda\psi = \left(\partial_x^4 + \frac{2}{3}az_x\partial_x^2 + \frac{2}{3}az_{xx}\partial_x + \frac{a}{18}\big(az_x^2 + 5z_{xxx} + 2z_t\big)\right)\psi, \tag{72}$$

$$\partial_t\psi = \left(2\partial_x^3 + az_x\partial_x + \frac{a}{2}z_{xx}\right)\psi, \tag{73}$$

with $z = cx$, $\lambda = \lambda_i$.

For $w = 0$, the system (3)–(4) degenerates into the KdV equation. Therefore, equation (70) possesses two types of solitary waves, respectively of forms (1) and (2). We further set $a = 1$. Applying the translation $z = \tilde{z} + cx$ on equation (70), we obtain the following dispersion relation for the equation in $\tilde{z}$:

$$F(\omega, k) \equiv 2\omega^2 + k^3\omega - k^6 - 3ck^4 - 2c^2k^2 = 0, \tag{74}$$

which is of second degree in ω. As in [9], we obtain two types of solitary waves, depending on the speed $v = -\omega/k$

$$v_+ = k^2 + c, \qquad v_- = -\tfrac{1}{2}k^2 - c,$$

where '+' and '−' correspond to the direction of propagation for $c = 0$.

The solitary wave propagating with the speed v_- is of the KdV type:

$$\begin{aligned} u_1^{(-)}(k) &\equiv z_{1,x}^{(-)}(k) = 6\partial_x^2 \log(1 + \mathrm{e}^{\theta_-}) + c \equiv 6k^2 \mathrm{sech}^2 \frac{\theta_-}{2} + c, \\ \theta_- &= kx + \omega_- t + \delta_- = kx + \tfrac{1}{2}k^3 t + ckt + \delta_-, \end{aligned} \tag{75}$$

while the solitary wave propagating with speed v_+ is of the KK type:

$$\begin{aligned} u_1^{(+)}(k) &\equiv z_{1,x}^{(+)}(k) = 6\partial_x^2 \log(1 + 2\mathrm{e}^{\theta_+} + A(k)\mathrm{e}^{2\theta_+}) + c, \\ A(k) &= \frac{3k^2 + 4c}{6k^2 + 4c}, \qquad \theta_+ = kx - k^3 t - ckt + \delta_+. \end{aligned} \tag{76}$$

At the level of the two-soliton

$$u_2(k_1, k_2) = 6\partial_x^2 \log f_2(k_1, k_2) + c, \tag{77}$$

the solution describing the interaction of two solitary waves of the same type, either with f_2 proportional to a 2×2 Wronskian, as in the KdV case,

$$f_2^{(--)}(k_1, k_2) = 1 + \mathrm{e}^{\theta_{1,-}} + \mathrm{e}^{\theta_{2,-}} + A_{12}\mathrm{e}^{\theta_{1,-}+\theta_{2,-}}, \qquad A_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{78}$$

or with f_2 proportional to a 2×2 Grammian, as in the KK case,

$$\begin{aligned} f_2^{(++)}(k_1, k_2) = {} & 1 + 2\,\mathrm{e}^{\theta_{1,+}} + 2\,\mathrm{e}^{\theta_{2,+}} + A_1\,\mathrm{e}^{2\theta_{1,+}} + A_2\,\mathrm{e}^{2\theta_{2,+}} \\ & + 4\frac{3\left(k_1^4 + k_2^4\right) + 4c\left(k_1^2 + k_2^2\right)}{(k_1 + k_2)^2\left(3k_1^2 + 3k_2^2 + 4c\right)}\mathrm{e}^{\theta_{1,+}+\theta_{2,+}} + A_{12}\mathrm{e}^{\theta_{1,+}+\theta_{2,+}}(A_1\,\mathrm{e}^{\theta_{1,+}} + A_2\,\mathrm{e}^{\theta_{2,+}}) \\ & + \frac{A_{12}^2}{4}A_1A_2\mathrm{e}^{2(\theta_{1,+}+\theta_{2,+})}, \qquad A_i = A(k_i), \quad \theta_{i,\pm} = \theta_\pm(k_i), \end{aligned} \tag{79}$$

cannot describe a resonant interaction because, for $k_1 \neq k_2$, $A_{12} > 0$.

However, when f_2 corresponds to the interaction of two solitary waves of different type:

$$f_2^{(+-)}(k_1, k_2) = 1 + 2\,\mathrm{e}^{\theta_{1,+}} + \mathrm{e}^{\theta_{2,-}} + A_1\,\mathrm{e}^{2\theta_{1,+}} + 2\tilde{A}_{12}\,\mathrm{e}^{\theta_{1,+}+\theta_{2,-}} + A_1\tilde{A}_{12}^2\,\mathrm{e}^{2\theta_{1,+}+\theta_{2,-}}, \tag{80}$$

$$\tilde{A}_{12} = \frac{6k_1^2 - 6k_1k_2 + 3k_2^2 + 4c}{6k_1^2 + 6k_1k_2 + 3k_2^2 + 4c}, \tag{81}$$

the condition $\tilde{A}_{12} = 0$ or $(\tilde{A}_{12})^{-1} = 0$ can be satisfied for $k_1 \neq k_2$. In the first case, for $c < 0$, solving the equation

$$6k_1^2 - 6k_1k_2 + 3k_2^2 + 4c = 0, \tag{82}$$

which implies the resonant relation

$$2\omega_+(k_1) - \omega_-(k_2) = \omega_-(2k_1 - k_2), \tag{83}$$

one obtains two different values for k_2:

$$k_2^\pm = -k_1 \pm \frac{1}{3}\sqrt{-9k_1^2 - 12c}, \qquad k_1^2 < -\frac{4}{3}c, \tag{84}$$

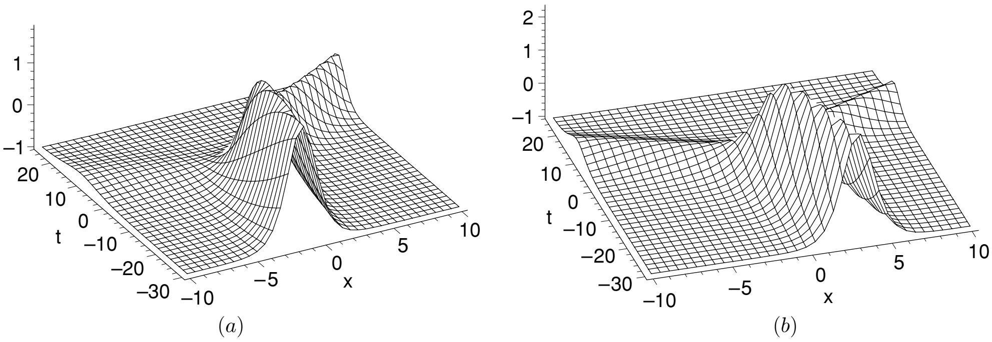

Figure 4. Decay of a KdV solitary wave (with speed $v_-(k_2^+)$) into a KdV- and a KK-type solitary waves: (*a*) $k_1 = 1/4$ ($v_-(k_2^+) > 0$), (*b*) $k_1 = 2/5$ ($v_-(k_2^+) < 0$) ($c = -1$).

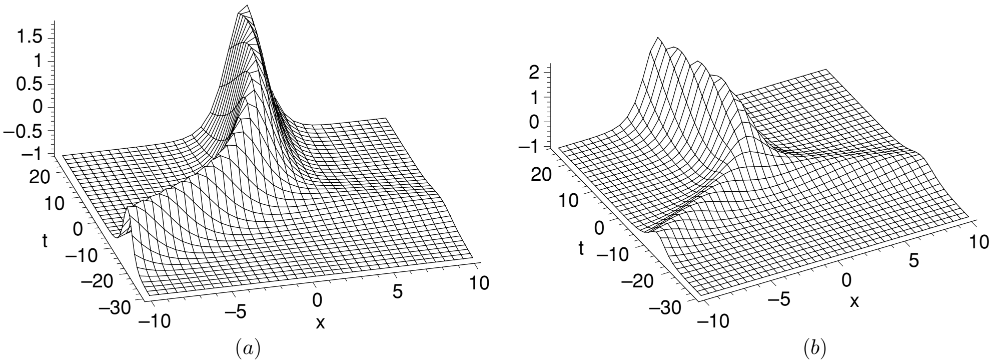

Figure 5. Fusion of a KdV- and a KK-type solitary waves into a KdV-type solitary wave (with speed $v_-(k_2^+)$): (*a*) $k_1 = 1/4$ ($v_-(k_2^+) > 0$), (*b*) $k_1 = 2/5$ ($v_-(k_2^+) < 0$) ($c = -1$).

with the restriction on k_1 ($A_1 > 0, i = 1, 2$)

$$k_1^2 < -\tfrac{2}{3}c. \tag{85}$$

In this case, expression (80) degenerates into

$$f_2^{(+-)}(k_1, k_2) = 1 + 2e^{\theta_{1,+}} + e^{\theta_{2,-}} + A_1 e^{2\theta_{1,+}}. \tag{86}$$

Therefore, performing an analysis similar to the bKK case, we obtain that in the case $k_2 = k_2^+$, the degenerate two-soliton solution describes the decay of a KdV-type solitary wave (with speed $v_-(k_2^+)$) into a KdV- and a KK-type solitary waves, propagating in opposite directions respectively with a speed $v_-(k_2^-) > 0$ and $v_+(k_1) < 0$ (see figure 4), while for $k_2 = k_2^- < 0$, it describes the reverse process of fusion (see figure 5).

For $(\tilde{A}_{12})^{-1} = 0$, corresponding to the ellipse

$$6k_1^2 + 6k_1k_2 + 3k_2^2 + 4c = 0, \qquad c < 0, \tag{87}$$

similar processes of fusion and decay may occur, which we do not describe in detail, due to the analogy with the previous case.

In conclusion, we show that the degeneracy of the two-soliton solution on curves (82) and (87) implies unusual resonance relations such as (83) and that the regularity of the resulting solution occurs only on parts of those ellipses. The asymptotic analysis shows that the resonant triad involves in this case two waves of the KdV type together with a third wave of the KK type.

5. Conclusion

The SK and KK equations are two integrable PDEs of fifth order in space and first order in time, which are in particular distinguishable by the analytical expression of their solitary waves, respectively of types (1) and (2).

The two-soliton solution of SK, built on a nonvanishing constant background, is known, when the resonance condition is satisfied, to degenerate into a regular solution describing a resonant triad of sech^2 waves. We here show that for KK, when the resonance condition is satisfied, the two-soliton degenerates into a solution developing a singularity at finite space and time.

The bSH equation (70) equivalent to the coupled system (3)–(4) describes overtaking and head-on collisions of solitons, which involve bell-shaped waves of types (1) and (2) according to their direction of propagation.

The soliton solutions of the bKK equation (20) equivalent to the system (5)–(6) involve only solitary waves of type (2), but their profile depends on the direction of propagation (see figure 1). For both equations, we here prove the existence of regular degenerate two-soliton solutions describing a resonant triad of bell-shaped waves. For the bKK equation, the three resonant waves of the KK type cannot possess the unusual profile of figure 1(*b*). For the bSH equation, the resonant triad includes two waves of the KdV type and one of the KK type.

The KK and bKK equations (7) and (20) are two different reductions of the (2 + 1)-dimensional CKP equation:

$$9z_{x_1,x_5} - 5z_{x_3x_3} + \Big(z_{x_1x_1x_1x_1x_1} + 15z_{x_1}z_{x_1x_1x_1} + 15\big(z_{x_1}\big)^3 - 5z_{x_1x_1,x_3} - 15z_{x_1}z_{x_3} + \tfrac{45}{4}\big(z_{x_1x_1}\big)^2\Big)_{x_1} = 0. \tag{88}$$

For this last equation, a detailed analysis of its soliton solutions which can be expressed in terms of Grammians [15] could provide an explanation to the appearance of resonant triad only in the bKK case corresponding to $z_{x_5} = 0 (t = 5x_3, x = x_1)$ and not in the KK case corresponding to $z_{x_3} = 0\ (t = 9x_5, x = x_1)$.

Moreover, it has been recently shown [16–19] that integrable equations in 2 + 1 dimensions belonging to the KP hierarchy, whose *N*-soliton solutions can be expressed in terms of Wronskians or Pfaffians, may describe a richer variety of exotic interactions than the (1 + 1)-dimensional soliton equations.

For those two reasons, we intend in a future work to extend the present analysis to the soliton equations in 2 + 1 dimensions of Grammian type which possess as reduction equations (70) and (20) here considered.

Acknowledgments

This work was supported by the project G.0359.03 of the FWO-Vlaanderen. I would like to thank Micheline Musette for the useful comments and discussions. The referees are acknowledged for their suggestions, which have clarified the presentation of this paper.

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